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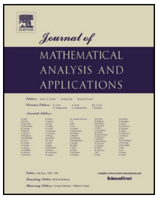

Regular Articles

Potential theory for quantum random walks associated with transition operation matrices

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ABSTRACT

We discuss the potential theory for quantum random walks associated with transition operation matrices. Both the well-known open quantum random walks and the unitary quantum random walks fall into this framework. The dual operation for a transition operation matrix defines a quantum Markov operator. We develop the potential theory for such quantum Markov operators, although much of the general theory has already been established. Using this approach, we characterize the reducibility/irreducibility and recurrence/transience of the quantum Markov operators associated with transition operation matrices. We then apply these results to open quantum random walks and unitary quantum random walks providing with some examples.

1. Introduction

In this paper, we discuss quantum random walks (QRWs) associated with transition operation matrices (TOMs), a framework introduced by Gudder [18]. TOMs generalize stochastic matrices in classical Markov chains in the sense that a TOM acts on vectors of trace-class operators while preserving the total trace, analogous to the preservation of total probability in classical Markov chains. Similarly to the backward and forward Kolmogorov equations in classical Markov chains, each TOM defines a dual operation, which we refer to as the quantum Markov operation associated with a TOM. The primary objective of this work is to develop the potential theory for such quantum Markov operations. We show that both well-known open

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quantum random walks (OQRWs) and unitary quantum random walks (UQRWs) fall within this framework. Consequently, basic characterizations such as reducibility/irreducibility and recurrence/transience for open or unitary QRWs can be studied through the general theory of QRWs associated with TOMs.

As noted in [6], OQRWs are not Markov chains in the classical sense. However, it is possible to associate a Markov chain with OQRWs via repeated measurements [6]. The key idea lies in the process of performing a measurement at each step of evolution, which gives rise to a classical state. The system is then reset to the observed state, and the next step proceeds from that state.

Since the introduction of QRWs–both unitary and open–there has been extensive research aimed at understanding their concrete probability distributions, asymptotic behavior, and related properties (see, for example, [14,23,24] for UQRWs and [5,6,25] for OQRWs, among others). To investigate the dynamical features of QRWs, the theory of quantum Markov chains has proven quite useful; see [11,10,22,26,27] for applications of this framework.

In this paper, we address the potential theory for quantum Markov operations associated with transition operation matrices, as initiated by Gudder [18]. On the one hand, this provides a new approach to defining quantum random walks that includes both OQRWs and UQRWs; on the other hand, it allows us to take advantage of the well-established properties of the general theory. Moreover, by embedding OQRWs and UQRWs within the framework of TOM, the study opens the door to cross-model comparisons and facilitates the application of quantum Markov operator techniques to problems in quantum measurement theory, quantum walks on graphs, and non-equilibrium quantum dynamics. The analysis of models on lattices and Cayley trees further demonstrates the versatility and applicability of the framework to both standard and more exotic quantum systems. In summary, this study advances both the mathematical foundations and the applicability of QRW theory by developing a robust, general framework rooted in potential theory, with implications for quantum probability, quantum computing, and open systems physics.

Quantum recurrence has been explored from various perspectives, often motivated by analogies with classical Markov chains. The study of recurrence in quantum processes has drawn attention across various mathematical frameworks. A notable analytic approach is found in [16], where the authors characterize recurrence for quantum Markov chains using Schur function techniques and splitting rules, drawing connections with Carathéodory functions and analytic function theory. In contrast, the present work offers a complementary operator-theoretic formulation, grounded in potential theory for quantum Markov operators derived from TOMs. This framework enables a rigorous treatment of recurrence, transience, and reducibility using tools from noncommutative probability and operator algebras. While both approaches address similar phenomena, the methods and scope differ significantly, suggesting potential for cross-fertilization between analytic and operator-theoretic perspectives.

While our work shares the same conceptual aim, namely, to characterize recurrence and transience in quantum stochastic processes our methodology is rooted in a different formalism. We employ the framework of TOMs, which naturally generalize classical stochastic matrices to the noncommutative setting. Using the associated quantum Markov operator $\mathcal{T}$, we develop a potential-theoretic characterization of recurrence based on the integrability of the operator series

$$\mathfrak{U}(x)[u] = \sum_{n=0}^{\infty} \langle u, \mathcal{T}^n(x) u \rangle,$$

and define recurrence in terms of the vanishing or divergence of such potentials when evaluated on projections.

In this sense, our results can be viewed as providing a *non-measurement-based, structural analogue* of the recurrence theory developed in [16]. Together, both approaches contribute to a deeper understanding of quantum recurrence, one from the *dynamical measurement* perspective, and the other from the *intrinsic operator-algebraic* behavior of quantum Markov evolutions.

We emphasize that the recurrence notion studied in this paper is a *non-measurement-based* one. It is defined through the unmonitored Heisenberg evolution generated by the quantum Markov operator $\mathcal{T}$ and the associated potential series

$$\mathfrak{U}(p)[u] = \sum_{n=0}^{\infty} \langle u, \mathcal{T}^n(p)u \rangle.$$

This should be distinguished from the *monitored* or measurement-based recurrence formalism, where after each step one projects onto the complement of the target site or subspace and studies first-return probabilities for the modified dynamics; see, for instance, [8,17]. Since monitoring changes the evolution itself, one should not expect the two notions to coincide in general. In Sections 4 and 5 we compare the two viewpoints in the concrete examples where the monitored behavior can be computed explicitly.

This paper is organized as follows. In Section 2, we introduce TOMs and the associated QRWs. In Section 3, we present the potential theory for QRWs associated with TOMs. Section 4 is devoted to applications to OQRWs on the one-dimensional integer lattice as well as on the Cayley trees, where we discuss reducibility/irreducibility and recurrence/transience. In Section 5, we consider UQRWs as another model and discuss some concrete properties by exhibiting suitable examples.

2. QRWs associated with TOMs

In this section we define QRWs through TOMs, which was developed by Gudder [18]. It turns out that the OQRWs and UQRWs fall into this regime.

Let Λ be a discrete set (intended to represent particle positions) and let $\mathcal{K} := l^2(\Lambda)$ be a separable Hilbert space with a canonical orthonormal basis $\{|i\rangle\}_{i\in\Lambda}$. The most common choice is to take the integer lattices $\Lambda = \mathbb{Z}^d$, but one can also consider arbitrary graphs. Each particle has an internal degree of freedom represented by a Hilbert space $\mathcal{H}$. Then we will consider the space $\mathcal{H} \otimes \mathcal{K}$. Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, the space of all bounded linear operators on $\mathcal{H} \otimes \mathcal{K}$, which is the weak closure of the space consisting of the elements of the form

$$x = \sum_{j\in\Lambda} x_j \otimes |j\rangle\langle j|, \quad x_j \in \mathcal{B}(\mathcal{H}),\ j \in \Lambda. \tag{2.1}$$

The TOMs are defined in the following way [18]. Denote by $\mathcal{I}(\mathcal{H})$ the ideal of trace class operators on $\mathcal{H}$. A TOM is a system of operators $\mathcal{E} = [\mathcal{E}^i_j]_{i,j\in\Lambda}$ on $\mathcal{I}(\mathcal{H})$ satisfying the following conditions:

(T1) For each pair i and j, $\mathcal{E}^i_j$ is a completely positive map on $\mathcal{I}(\mathcal{H})$,
(T2) It holds that $\mathrm{Tr}(\sum_i \mathcal{E}^i_j(D)) = \mathrm{Tr}(D)$ for all $j \in \Lambda$ and $D \in \mathcal{I}(\mathcal{H})$.

Given a TOM $\mathcal{E} = [\mathcal{E}^i_j]_{i,j\in\Lambda}$, we define a **quantum random walk associated with** $\mathcal{E}$ by

$$\mathcal{M}^{(\mathcal{E})}(\rho) := \sum_i \Big(\sum_j \mathcal{E}^i_j(\rho_j) \Big) \otimes |i\rangle\langle i|, \quad \rho = \sum_i \rho_i \otimes |i\rangle\langle i|. \tag{2.2}$$

In quantum mechanics, the Schrödinger picture and the Heisenberg picture are dual concepts that provide different perspectives on time evolution: one from the viewpoint of states and the other from that of observables. Similarly, we introduce a dual map ${\mathcal{E}^i_j}^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ so that

$$\mathrm{Tr}(\mathcal{E}^i_j(D)x) = \mathrm{Tr}(D{\mathcal{E}^i_j}^*(x)), \quad D \in \mathcal{I}(\mathcal{H}),\ x \in \mathcal{B}(\mathcal{H}). \tag{2.3}$$

By the condition (T2), it follows that

$$\sum_i \mathcal{E}_j^{i*}(\mathbb{1}) = \mathbb{1}. \tag{2.4}$$

In fact, notice that the predual of $\mathcal{B}(\mathcal{H})$ is $\mathcal{I}(\mathcal{H})$. So, let $D \in \mathcal{I}(\mathcal{H})$ be any element. Then,

$$\begin{aligned}\mathrm{Tr}\left(D\sum_i \mathcal{E}_j^{i*}(\mathbb{1})\right) &= \mathrm{Tr}\left(\sum_i \mathcal{E}_j^{i}(D)\right)\\ &= \mathrm{Tr}(D),\end{aligned}$$

where we have used (T2). Since $D \in \mathcal{I}(\mathcal{H})$ is arbitrary, it shows (2.4). Given a TOM, using the above dual map we now introduce the Markov operator associated with $\mathcal{E}$ on $\mathcal{A}$: for any $x = \sum_{j\in\Lambda} x_j \otimes |j\rangle\langle j| \in \mathcal{A}$, define

$$\mathcal{T}^{(\mathcal{E})}(x) := \sum_j \left(\sum_i \mathcal{E}_j^{i*}(x_i)\right) \otimes |j\rangle\langle j|. \tag{2.5}$$

By (2.4), $\mathcal{T}^{(\mathcal{E})}$ preserves the identity in $\mathcal{A}$ and we call $\mathcal{T}^{(\mathcal{E})}$ the **Markov operator on $\mathcal{A}$ associated with a TOM $\mathcal{E}$**. Whenever there is no danger of confusion we will simply write $\mathcal{M}$ and $\mathcal{T}$ for $\mathcal{M}^{(\mathcal{E})}$ and $\mathcal{T}^{(\mathcal{E})}$, respectively.

The n-fold repetition of $\mathcal{T}$ yields

$$\mathcal{T}^n(x) = \sum_j \left(\sum_{i_{n-1}} \cdots \sum_{i_1}\sum_i \mathcal{E}_j^{i_{n-1}*} \circ \cdots \circ \mathcal{E}_{i_2}^{i_1*} \circ \mathcal{E}_{i_1}^{i\,*}(x_i)\right) \otimes |j\rangle\langle j|. \tag{2.6}$$

The nth power of $\mathcal{M}$ will be further simplified as $\rho^{(n)} := \mathcal{M}^n(\rho^{(0)})$. The fact that $\mathcal{T}$ is the dual map of $\mathcal{M}$ is shown by the following proposition.

Proposition 2.1. *For any initial state $\rho^{(0)} = \sum_i \rho_i^{(0)} \otimes |i\rangle\langle i|$ and $x = \sum_i x_i \otimes |i\rangle\langle i| \in \mathcal{A}$, it holds for any $n \geq 0$ that*

$$\rho^{(0)}(\mathcal{T}^n(x)) = \rho^{(n)}(x).$$

Proof. Noticing that density operators and states are used interchangeably, we see from (2.5) and (2.3) that

$$\begin{aligned}\rho^{(0)}(\mathcal{T}(x)) &= \mathrm{Tr}(\rho^{(0)}\mathcal{T}(x))\\ &= \sum_j \mathrm{Tr}\left(\rho_j^{(0)}\sum_i \mathcal{E}_j^{i*}(x_i)\right)\\ &= \sum_i \mathrm{Tr}\left(\sum_j \mathcal{E}_j^{i}(\rho_j^{(0)})x_i\right)\\ &= \sum_i \mathrm{Tr}\left(\rho_i^{(1)}x_i\right)\\ &= \rho^{(1)}(x).\end{aligned}$$

Repeating this process n times, we obtain the result. □

3. Potential theory for QRWs associated with TOMs

In this section we discuss the potential theory for the quantum Markov operator $\mathcal{T}$ associated with a TOM. But, before we begin, we make a few remarks. There is another way to define a quantum Markov chain by using the transition expectations, which was initiated by Accardi in the 1970s [1] and further developed subsequently, see for instance [2,3]. Many applications of quantum Markov chains have since emerged. To mention a few papers related to the present work, one can refer to [2,3,9–11]. The backward and forward Markov transition operators play a crucial role to define a quantum Markovian dynamics. In the present model, however, we found that this approach was not appropriate for investigating the potential theory of OQRWs; therefore, we have proposed the quantum Markov operator described in the previous section.

Now we fix a TOM $\mathcal{E} = [\mathcal{E}_j^i]$ on $\mathcal{I}(\mathcal{H})$ and write $\mathcal{M}$ and $\mathcal{T}$ for $\mathcal{M}^{(\mathcal{E})}$ and $\mathcal{T}^{(\mathcal{E})}$, respectively. In the sequel, we adopt the definitions from [9,28] and refer the reader to these papers for further details. For all positive operator $x \in \mathcal{A}$, let us define

$$\begin{aligned} \mathrm{Dom}(\mathfrak{U}(x)) &= \{u \in \mathcal{H} \otimes \mathcal{K} : \sum_{n\geq 0} \langle u, \mathcal{T}^n(x)u\rangle < \infty\}, \\ \mathfrak{U}(x)[u] &:= \sum_{n\geq 0} \langle u, \mathcal{T}^n(x)u\rangle, \quad u \in \mathrm{Dom}(\mathfrak{U}(x)). \end{aligned} \tag{3.1}$$

Definition 3.1. A positive operator $x \in \mathcal{A}$ such that $\mathrm{Dom}(\mathfrak{U}(x))$ is dense is called **integrable**. For an integrable x, we denote by $\mathcal{U}(x)$ the self-adjoint operator which represents the quadratic form $\mathfrak{U}(x)$. A positive operator $y \in \mathcal{A}$ is called a **potential** if there exists an integrable $x \in \mathcal{A}$ such that $y = \mathcal{U}(x)$.

Definition 3.2. A projection p is **recurrent** if, for all u in the range of p either $u \notin \mathrm{Dom}(\mathfrak{U}(p))$ or $u \in \mathrm{Dom}(\mathfrak{U}(p))$ and $\mathfrak{U}(p)[u] = 0$. $\mathcal{T}$ is called **recurrent** if every projection is recurrent.

From now on, we review some basic properties relevant to the potential theory for QRWs associated with TOMs. Most of the results were shown in [9,13]. We will only recall the results without proofs. In fact, all the proofs of the following results can be given exactly in the same way that are given in [9] by replacing $\mathcal{T} \circ \mathcal{T}'$ there with $\mathcal{T}$ here. For the details and more results we refer to Sections 3 and 4 of [9], and also [13,28].

Definition 3.3. A densely defined self-adjoint operator y on $\mathcal{H} \otimes \mathcal{K}$ is said to be **affiliated** with the von Neumann algebra $\mathcal{A}$ if all spectral projections of y belong to $\mathcal{A}$. Equivalently, $vy \subset yv$ for every unitary $v \in \mathcal{A}'$, the commutant of $\mathcal{A}$.

Proposition 3.4. *([9, Proposition 1]) For all integrable $x \in \mathcal{A}$, the self-adjoint operator $\mathcal{U}(x)$ is affiliated with $\mathcal{A}$.*

Definition 3.5. A self-adjoint element $x \in \mathcal{A}$ is called **$\mathcal{T}$-subharmonic** (resp. **$\mathcal{T}$-superharmonic**) if $\mathcal{T}(x) \geq x$ (resp. $\mathcal{T}(x) \leq x$).

Theorem 3.6. *([9, Theorem 1]) An element $y \in \mathcal{A}$ is a potential if and only if it is $\mathcal{T}$-superharmonic and $\mathcal{T}^n(y)$ converges strongly to 0 as $n \to \infty$.*

Theorem 3.7. *([9, Theorem 3]) For all integrable $x \in \mathcal{A}$, the contraction*

$$y = \mathcal{U}(x)(\mathbb{1} + \mathcal{U}(x))^{-1}$$

is $\mathcal{T}$-superharmonic and $\mathcal{T}^m(y)$ converges strongly to 0 as $m \to \infty$. In particular y is a potential.

We adopt the definition of transience from [9]:

Definition 3.8. A projection $p \in \mathcal{A}$ is called **transient** if there exists a family $(p_i)_{i\in I}$ of projections such that $p \le \bigvee_{i\in I} p_i$ and $\mathcal{U}(p_i)$ is bounded for all i. $\mathcal{T}$ is called **transient** if the identity $\mathbb{1}$ is transient.

Definition 3.9. $\mathcal{T}$ is called **irreducible** if there exists no non-trivial projection p which is $\mathcal{T}$-subharmonic, i.e., $\mathcal{T}(p) \ge p$. Otherwise, it is called **reducible**.

Remark 3.10. Once $\mathcal{T}$ is identity preserving, it is easy to see that the definition of irreducibility is equivalent to saying that there exists no non-trivial projection p which is $\mathcal{T}$-superharmonic. In [9], the definition of irreducibility is given this way.

Theorem 3.11. *([9, Theorem 4]) If $\mathcal{T}$ is irreducible, then it is either recurrent or transient.*

To apply the above general properties of potential theory to our model of QRWs associated with TOMs, we need slightly more details. For quantum Markov semigroup, subharmonicity is also characterized by the following theorem.

Theorem 3.12. *([28, Theorem 2]) A projection p is $\mathcal{T}$-subharmonic if and only if*

$$p\mathcal{T}(a)p = p\mathcal{T}(pap)p, \quad \forall\, a \in \mathcal{A}. \tag{3.2}$$

Remark 3.13. By induction, (3.2) is equivalent to saying that

$$p\mathcal{T}^n(a)p = p\mathcal{T}^n(pap)p, \quad \forall\, a \in \mathcal{A},\ \forall\, n \ge 0. \tag{3.3}$$

The irreducibility of OQRWs was also discussed in [7] as well as in [10]. So, it is worth investigating the relationship between the definitions in [7,10] and in Definition 3.9. First, it was shown in [10] that the definitions of irreducibility in [7] and [10] are the same. Motivated by the work in [10], we can find an equivalent condition for the reducibility/irreducibility for the quantum Markov operator $\mathcal{T}$ associated with a TOM.

Theorem 3.14. *Let $\mathcal{E} = [\mathcal{E}^i_j]$ be a TOM and $\mathcal{M}^{(\mathcal{E})}$ and $\mathcal{T}^{(\mathcal{E})}$ be the QRW and quantum Markov operator, respectively, associated with $\mathcal{E}$. Then, $\mathcal{T}^{(\mathcal{E})}$ is irreducible in the sense of Definition 3.9 if and only if there exists no non-trivial projection $p = \sum_{j\in\Lambda} p_j \otimes |j\rangle\langle j|$ such that*

$$p\rho^{(n)}p = \rho^{(n)}, \quad \forall\, \rho^{(0)} \text{ with } p\rho^{(0)}p = \rho^{(0)}, \tag{3.4}$$

where $\rho^{(n)} = \big(\mathcal{M}^{(\mathcal{E})}\big)^n(\rho^{(0)})$.

Proof. We suppress the $\mathcal{E}$-dependence in the notation, as usual. Suppose that there exists a non-trivial projection p satisfying (3.4). Let ρ be an arbitrary state (density matrix) and take $\rho^{(0)} = p\rho p$. Then, by using Proposition 2.1, for any $a \in \mathcal{A}$ and $n \ge 0$, since $p\rho^{(n)}p = \rho^{(n)}$, on the one hand we have

$$\begin{aligned}\rho^{(n)}(a) &= \rho^{(0)}(\mathcal{T}^n(a))\\ &= p\rho p(\mathcal{T}^n(a))\\ &= \rho(p\mathcal{T}^n(a)p).\end{aligned} \tag{3.5}$$

But, on the other hand,

$$\begin{aligned} p\rho^{(n)}p(a) &= \rho^{(n)}(pap) \\ &= \rho^{(0)}(\mathcal{T}^n(pap)) \\ &= p\rho p(\mathcal{T}^n(pap)) \\ &= \rho(p\mathcal{T}^n(pap)p). \end{aligned} \tag{3.6}$$

The right hand sides of (3.5) and (3.6) are the same for all ρ. Therefore, we conclude that

$$p\mathcal{T}^n(a)p = p\mathcal{T}^n(pap)p.$$

By Theorem 3.12 p is $\mathcal{T}$-subharmonic. Therefore, $\mathcal{T}$ is reducible.

Conversely, suppose that $\mathcal{T}$ is reducible. By the equations (3.5) and (3.6) in the reversed order, we arrive at the conclusion

$$\rho^{(n)}(a) = p\rho^{(n)}p(a), \quad a \in \mathcal{A} \text{ and } \rho^{(0)} \text{ with } p\rho^{(0)}p = \rho^{(0)}.$$

The proof is completed. □

Since we have defined TOMs on $\mathcal{I}(\mathcal{H})$, it is worthwhile to derive sufficient conditions for the reducibility of $\mathcal{T}$ based solely on the properties of the TOMs.

Theorem 3.15. *Let $\mathcal{E} = [\mathcal{E}^i_j]$ be a TOM on $\mathcal{I}(\mathcal{H})$. Suppose that h is a non-trivial projection on $\mathcal{H}$ such that*

$$h\mathcal{E}^i_j(D)h = \mathcal{E}^i_j(D), \quad \textit{for all } i, j \in \Lambda \textit{ and } D \in \mathcal{I}(H).$$

Then $\mathcal{T}$ is reducible.

Proof. We first show that for the projection h satisfying the property in the statement of the theorem

$$\sum_i \mathcal{E}^{i\,*}_j(h) = \mathbb{1} \text{ for all } j \in \Lambda. \tag{3.7}$$

In fact, for any $D \in \mathcal{I}(\mathcal{H})$,

$$\begin{aligned} \operatorname{Tr}\Big(D\sum_i \mathcal{E}^{i\,*}_j(h)\Big) &= \operatorname{Tr}\Big(\sum_i \mathcal{E}^i_j(D)h\Big) \\ &= \operatorname{Tr}\Big(h\sum_i \mathcal{E}^i_j(D)h\Big) \\ &= \operatorname{Tr}\Big(\sum_i \mathcal{E}^i_j(D)\Big) \\ &= \operatorname{Tr}(D). \end{aligned}$$

Since $D \in \mathcal{I}(\mathcal{H})$ is arbitrary, the claim is proved. Now let p be a projection defined by

$$p = \sum_j h \otimes |j\rangle\langle j|.$$

Obviously p is a non-trivial projection in $\mathcal{A}$. Then, by (3.7)

$$\begin{aligned}\mathcal{T}(p) &= \sum_j \sum_i \mathcal{E}_j^{i^*}(h) \otimes |j\rangle\langle j| \\ &= \sum_j \mathbb{1} \otimes |j\rangle\langle j| \geq p.\end{aligned} \tag{3.8}$$

Thus, p is $\mathcal{T}$-subharmonic and so $\mathcal{T}$ is reducible. □

4. OQRWs

In this section, we consider OQRWs as special cases of QRWs associated with TOMs.

4.1. Definition of OQRWs

Let us briefly recall the definition of OQRWs. For each pair i, j it is associated a bounded linear operator B_j^i on $\mathcal{H}$ such that

$$\sum_i B_j^{i*} B_j^i = \mathbb{1}. \tag{4.1}$$

Mostly we will deal with the cases where the sum addresses only finite number, but if it is an infinite sum we assume that the series is strongly convergent. The operators B_j^i act on $\mathcal{H}$ only, we dilate them as operators on $\mathcal{H} \otimes \mathcal{K}$ by defining

$$M_j^i = B_j^i \otimes |i\rangle\langle j|\,.$$

It is clear that

$$\sum_{i,j} M_j^{i^*} M_j^i = \mathbb{1}.$$

Therefore, the operators $(M_j^i)_{i,j}$ define a completely positive mapping

$$\mathcal{M}(\rho) = \sum_i \sum_j M_j^i \, \rho \, M_j^{i^*} \tag{4.2}$$

on the state space of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Consider density matrices in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which take the form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|. \tag{4.3}$$

Given an initial state of this form, the *open quantum random walk* is defined by the map,

$$\mathcal{M}(\rho) = \sum_i \Big(\sum_j B_j^i \rho_j B_j^{i*} \Big) \otimes |i\rangle\langle i|. \tag{4.4}$$

As the following remark shows the OQRWs always can be understood as QRWs associated with some TOMs.

Remark 4.1. Consider an OQRW. Given generating operators $\{B_j^i : i, j \in \Lambda\}$ satisfying (4.1), define for each pair $i, j \in \Lambda$

$$\mathcal{E}_j^i(D) := B_j^i D B_j^{i^*}, \quad D \in \mathcal{I}(\mathcal{H}). \tag{4.5}$$

Clearly, $\mathcal{E} = [\mathcal{E}_j^i]_{i,j \in \Lambda}$ is a TOM on $\mathcal{I}(\mathcal{H})$.

4.2. Classical Markov chain

Before going further, we discuss the classical Markov chain on the countable state space. In several papers it was shown that the classical Markov chain could be viewed as an OQRW (hence a QRW associated with a TOM in the present terminology), and also as a quantum Markov chain [2,3,6,10,9]. We follow [10, Section 5.2].

Let $\mathcal{H} = \mathbb{C}$ and $\mathcal{K} = l^2(\Lambda)$. Then $\mathcal{H} \otimes \mathcal{K} \cong l^2(\Lambda)$ and $\mathcal{M} = \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \cong l^\infty(\Lambda)$, where any element of $l^\infty(\Lambda)$ acts as a multiplication operator on the Hilbert space $l^2(\Lambda)$. Let $P = (P(i,j))_{i,j\in\Lambda}$ be a stochastic matrix on Λ. For each $i, j \in \Lambda$, let U^i_j be a unitary operator on $\mathcal{H}$, or a complex number of modulus 1. Define

$$B^i_j := \sqrt{P(j,i)}U^i_j, \quad i, j \in \Lambda.$$

We see that

$$\sum_i B^{i\,*}_j B^i_j = \mathbb{1}, \quad j \in \Lambda.$$

Now we construct a TOM $\mathcal{E} = [\mathcal{E}^i_j]$ on $\mathcal{I}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \equiv \mathbb{C}$ by (4.5). We easily compute for $D, x \in \mathcal{B}(\mathcal{H})$ (surely D and x are nothing but some complex numbers since $\mathcal{H} = \mathbb{C}$ is a one-dimensional space)

$$\mathcal{E}^i_j(D) = P(j,i)D, \quad \mathcal{E}^{i\,*}_j(x) = P(j,i)x. \tag{4.6}$$

Notice that the algebra $\mathcal{A}$ consisting of the operators $x = \sum_j x_j \otimes |j\rangle\langle j|$, with $(x_j)_{j\in\Lambda}$ a bounded sequence in $\mathbb{C}$, is a commutative algebra. The following proposition shows that the operator $\mathcal{T}$ recovers the (backward) dynamics of classical Markov chain.

Proposition 4.2. *For any $x = (x_j)_{j\in\Lambda} \in l^2(\Lambda)$, it holds that*

$$\mathcal{T}^n(x) = P^n x, \quad n \geq 0,$$

where $(P^n x)_i = \sum_j P^n(i,j)x_j$.

Proof. Using (2.6) and (4.6) we compute

$$\begin{aligned}\mathcal{T}^n(x) &= \sum_j \Big(\sum_{i_{n-1}} \cdots \sum_{i_1} \sum_i P(j, i_{n-1}) \cdots P(i_1, i) x_i \Big) \otimes |j\rangle\langle j| \\ &= \sum_j (P^n x)_j \otimes |j\rangle\langle j|.\end{aligned}$$

The last term is $P^n x = ((P^n x)_j)_{j\in\Lambda}$, viewed as a sequence. □

Now, fix $j \in \Lambda$ and let $p_j \equiv \delta_j \in l^\infty(\Lambda)$ be the projection onto the site j. Any element $u \in l^2(\Lambda)$ in the range of p_j is a form of $u = c\delta_j$ for some constant c. Then,

$$\langle u, \mathcal{T}^n(p_j)\, u\rangle = |c|^2 P^n(j,j).$$

Therefore we see that

$$\sum_{n=0}^\infty \langle u, \mathcal{T}^n(p_j)\, u\rangle = |c|^2 \sum_{n=0}^\infty P^n(j,j). \tag{4.7}$$

Recall that in the classical Markov chain a state (site) $j \in \Lambda$ is recurrent if and only if $\sum_{n=0}^{\infty} P^n(j,j) = \infty$ (see for example [15, Section 6.2]). Thus we conclude that if $c \neq 0$, then

$$\sum_{n\geq 0} \langle u, \mathcal{T}^n(p_j)\, u\rangle = \infty \text{ if and only if } j \text{ is recurrent (in the classical sense).}$$

We summarize this in the following proposition.

Proposition 4.3. *For the classical Markov chain on a set Λ with a stochastic matrix P, any state (site) $j \in \Lambda$ is recurrent (in the classical sense) if and only if the projection p_j is recurrent in the sense of Definition 3.2.*

4.3. Properties

We first examine the sufficient condition for reducibility that is specific to OQRWs.

Proposition 4.4. *Suppose there is a nontrivial projection h in $\mathcal{B}(\mathcal{H})$ such that $hB_j^i = B_j^i$ for all B_j^i's. Then, the associated quantum Markov operator $\mathcal{T}$ is reducible.*

Proof. Let $D \in \mathcal{I}(\mathcal{H})$ be any trace class operator. Then

$$\begin{aligned} h\mathcal{E}_j^i(D)h &= hB_j^i D {B_j^i}^* h \\ &= hB_j^i D (hB_j^i)^* \\ &= B_j^i D {B_j^i}^* = \mathcal{E}_j^i(D). \end{aligned}$$

We conclude by Theorem 3.15 that $\mathcal{T}$ is reducible. □

We call a subset $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$ is self-adjoint if $\mathcal{C}^* = \mathcal{C}$, where $\mathcal{C}^* = \{y^* : y \in \mathcal{C}\}$.

Theorem 4.5. *Suppose that the set of operators $\{B_j^i : i,j \in \Lambda\}$ is self-adjoint. Then, a projection $p = \sum_j h \otimes |j\rangle\langle j|$ is $\mathcal{T}$-subharmonic if and only if $h \in \{B_j^i : i,j \in \Lambda\}'$. In this case p is $\mathcal{T}$-harmonic: $\mathcal{T}(p) = p$.*

Proof. Suppose that p is $\mathcal{T}$-subharmonic. Then,

$$\begin{aligned} \mathcal{T}(p) &= \sum_j \left(\sum_i {B_j^i}^* h B_j^i \right) \otimes |j\rangle\langle j| \\ &\geq \sum_j h \otimes |j\rangle\langle j|. \end{aligned}$$

This means that $\sum_i {B_j^i}^* h B_j^i \geq h$ for all $j \in \Lambda$. Multiplying both sides of the equation by h from the left and right, we obtain

$$\sum_i h{B_j^i}^* h B_j^i h \geq h, \quad \forall\, j \in \Lambda.$$

Hence,

$$\sum_i (h^\perp B_j^i h)^* (h^\perp B_j^i h) = \sum_i h{B_j^i}^* h^\perp B_j^i h = \sum_i h{B_j^i}^* (\mathbb{1} - h) B_j^i h \leq p - p = 0.$$

Therefore, $h^{\perp}B_j^i h = 0$, or $B_j^i h = hB_j^i h$ for all $i, j \in \Lambda$. Now, by the assumption, $B_j^i = {B_k^l}^*$ for some B_k^l and hence

$$hB_j^i = (B_k^l h)^* = (hB_k^l h)^* = h{B_k^l}^* h = hB_j^i h = B_j^i h.$$

It shows that $h \in \{B_j^i : i, j \in \Lambda\}'$. Conversely, suppose that $h \in \{B_j^i : i, j \in \Lambda\}'$. Then,

$$\begin{aligned}\mathcal{T}(p) &= \sum_j \left(\sum_i {B_j^i}^* h B_j^i\right) \otimes |j\rangle\langle j| \\ &= \sum_j \left(\sum_i {B_j^i}^* B_j^i h\right) \otimes |j\rangle\langle j| \\ &= \sum_j h \otimes |j\rangle\langle j| \\ &= p,\end{aligned}$$

that is p is $\mathcal{T}$-harmonic, and of course $\mathcal{T}$-subharmonic. This completes the proof. □

Remark 4.6. A similar result as above for the quantum dynamical system appears in [12, Corollary III.1].

4.4. Examples

In this section we consider some concrete examples.

4.4.1. OQRWs on $\mathbb{Z}$

First we revisit [10, Example 5.1]. We consider a one-dimensional OQRW in $\mathbb{Z}$. Let $\mathcal{H} = \mathbb{C}^2$ and let $B, C \in \mathcal{B}(\mathcal{H})$ be 2×2 matrices such that $B^*B + C^*C = \mathbb{1}$. We define the OQRW as follows:

$$B_i^{i-1} = B \text{ and } B_i^{i+1} = C$$

for all $i \in \mathbb{Z}$, and $B_j^i = 0$ otherwise.

(i) Recurrence. Let us take

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad h = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Notice that $hB = B$, $hC = C$. By Theorem 3.15, $\mathcal{T}$ is reducible. Now by the computation in (3.8), $\mathcal{T}(p) = \mathbb{1}$ and hence $\mathcal{T}^n(p) = \mathbb{1}$ for all $n \geq 1$. h is a rank 1 projection whose range is generated by the vector $\psi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus any vector in the range of p is of the form

$$u = \sum_{j \in \mathbb{Z}} (c_j \psi) \otimes |j\rangle,$$

where $(c_j)_{j \in \mathbb{Z}}$ is a sequence of complex numbers such that $\sum_{j \in \mathbb{Z}} |c_j|^2 < \infty$. We see that

$$\sum_{n \geq 0} \langle u, \mathcal{T}^n(p) u\rangle = \infty$$

if $u \neq 0$. We conclude that the projection p is recurrent.

(ii) Transience. Here we consider a model where there are transient projections. Again we consider a one-dimensional OQRW. Consider the canonical projections in $\mathcal{H} = \mathbb{C}^2$:

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.8}$$

Define

$$B_j^i = \begin{cases} B := P_1, & i = j-1, \\ C := P_2, & i = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

We immediately see that

$$B^* P_1 B = P_1, \quad B^* P_2 B = 0, \quad C^* P_1 C = 0, \quad C^* P_2 C = P_2. \tag{4.9}$$

Let $p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$. Then, we obtain

$$\mathcal{T}^n(p_0) = P_1 \otimes |n\rangle\langle n| + P_2 \otimes |-n\rangle\langle -n|.$$

Therefore, for any $u = \sum_{j\in\mathbb{Z}} u_j \otimes |j\rangle \in \mathcal{H} \otimes l^2(\mathbb{Z})$,

$$\begin{aligned} \sum_{n\geq 0} \langle u, \mathcal{T}^n(p_0) u\rangle &= \sum_{n\geq 0} \langle u, (P_1 \otimes |n\rangle\langle n| + P_2 \otimes |-n\rangle\langle -n|) u\rangle \\ &= \|u_0\|^2 + \sum_{n=1}^{\infty} \left(\|P_1 u_n\|^2 + \|P_2 u_{-n}\|^2\right) \\ &\leq \|u\|^2. \end{aligned}$$

We conclude that $u \in \mathrm{Dom}(\mathfrak{U}(p_0))$ and moreover, $\mathcal{U}(p_0) = \sum_{n=0}^{\infty} (P_1 \otimes |n\rangle\langle n| + P_2 \otimes |-n\rangle\langle -n|)$, which is a projection and hence $\|\mathcal{U}(p_0)\| = 1$. The projection p_0 is transient. Now for each $k \in \mathbb{N}$, let $p_{[k]}$ denote the projection of $\mathcal{H} \otimes l^2(\mathbb{Z})$ onto the set of local vectors supported on the set $[-k, k] \subset \mathbb{Z}$. By a similar observation as above, given a projection $p_{[k]}$, we easily see that

$$\mathcal{T}^{k+n}(p_{[k]}) = \sum_{l=n}^{k+n} (P_1 \otimes |l\rangle\langle l| + P_2 \otimes |-l\rangle\langle -l|).$$

Therefore, we compute for any $u \in \mathcal{H} \otimes l^2(\mathbb{Z})$,

$$\sum_{n=0}^{\infty} \langle u, \mathcal{T}^n(p_{[k]}) u\rangle \leq (k+1)\|u\|^2.$$

Thus, $\mathfrak{U}(p_{[k]})$ is a bounded operator. Notice that $\bigvee_{n\geq 1} p_{[n]} = \mathbb{1}$. We conclude that the identity projection $\mathbb{1}$ is transient and hence the Markov operator $\mathcal{T}$ is transient.

(iii) Recurrence and Transience. A model similar to the following was discussed in [10, Example 5.4]. Let $U = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ be a (real) unitary matrix such that $|\alpha| \leq 1$, $|\beta| \leq 1$, and $\alpha^2 + \beta^2 = 1$. Let

$$B = \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -\beta \\ 0 & \alpha \end{bmatrix}.$$

The OQRW on $\mathbb{Z}$ generated by the above B and C was shown to be irreducible in [10]. Now we discuss the potential theory. The notations of p_0, P_1, and P_2 are as in the previous subsection. We compute:

$$B^* P_1 B = |\alpha|^2 P_1, \quad B^* P_2 B = |\beta|^2 P_1, \quad C^* P_1 C = |\beta|^2 P_2, \quad C^* P_2 C = |\alpha|^2 P_2. \tag{4.10}$$

We start by considering some typical cases.

Case 1: $\alpha = 1$, $\beta = 0$.
In this case the operations in (4.10) are the same as those in (4.9). Therefore, the Markov operator $\mathcal{T}$ is transient.

Case 2: $\alpha = 0$, $\beta = 1$.
In this case, for $p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$ the dynamics is given by

$$\mathcal{T}^n(p_0) = \begin{cases} \mathbb{1} \otimes |0\rangle\langle 0|, & n \text{ even}, \\ P_1 \otimes |1\rangle\langle 1| + P_2 \otimes |-1\rangle\langle -1|, & n \text{ odd}. \end{cases}$$

Therefore, if u is in the range of p_0, that is, $u = u_0 \otimes |0\rangle$ with $u_0 \in \mathcal{H}$, we get $\mathfrak{U}(p_0)[u] = \infty$ if $u_0 \neq 0$. We conclude that p_0 is recurrent.

Case 3: $\alpha = \beta = 1/\sqrt{2}$.
In this case the dynamics is quite similar to the classical simple random walk. But there is a subtlety in the computation. The dynamics for the first few time steps are as follows:

$$\begin{aligned} \mathcal{T}(p_0) &= P_1 \otimes |1\rangle\langle 1| + P_2 \otimes |-1\rangle\langle -1|, \\ \mathcal{T}^2(p_0) &= \frac{1}{2} P_1 \otimes |2\rangle\langle 2| + \frac{1}{2} \mathbb{1} \otimes |0\rangle\langle 0| + \frac{1}{2} P_2 \otimes |-2\rangle\langle -2|, \\ p_0 \mathcal{T}^4(p_0) p_0 &= \left(\frac{1}{2^2} + \frac{1}{2^3} \right) \mathbb{1} \otimes |0\rangle\langle 0|. \end{aligned}$$

At each even time, the restriction $p_0 \mathcal{T}^{2n}(p_0) p_0$ is a constant multiple of $\mathbb{1} \otimes |0\rangle\langle 0|$. A weight is contributed to the coefficient from each random walk path starting from 0 and arriving at 0 after $2n$ time. The weights depend on the paths but they are always greater than those of simple random walk, and so is the total weight. For example, the probability of returning at the origin of a simple random walk at time 4 is $\binom{4}{2} \frac{1}{2^4} = \frac{3}{8}$ but the coefficient of $p_0 \mathcal{T}^4(p_0) p_0$ is $\frac{1}{2^2} + \frac{1}{2^3} = \frac{5}{12} > \frac{3}{8}$. So we can proceed as follows. For a vector $u = u_0 \otimes |0\rangle \in \mathrm{Ran}(p_0)$,

$$\begin{aligned} \sum_{n \geq 0} \langle u, \mathcal{T}^{2n}(p_0) u \rangle &= \sum_{n \geq 0} \langle u, p_0 \mathcal{T}^{2n}(p_0) p_0 u \rangle \\ &> \sum_{n \geq 0} \binom{2n}{n} \frac{1}{2^{2n}} \|u_0\|^2. \end{aligned}$$

By the Stirling's formula we have $\binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$. Therefore, the above summation diverges unless $u_0 = 0$. We conclude that the projection p_0 is recurrent.

The general case: In the general case, the subtlety of path dependence increases further. Fortunately, Fourier analysis helps in this situation. Here, we state the result and postpone its proof in the Appendix.

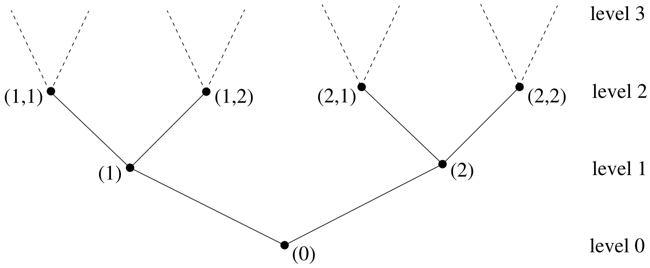

Fig. 1. The first levels of Γ_+^2.

Proposition 4.7. *For the OQRW that satisfy (4.10), for all values of α and β such that $\alpha^2+\beta^2=1$, except the case $\alpha=1$ and $\beta=0$, the projections $p_j=\mathbb{1}\otimes|j\rangle\langle j|$, $j\in\mathbb{Z}$, are recurrent for the Markov operator $\mathcal{T}$.*

Remark 4.8. The lesson that the above proposition gives us is the difference between OQRWs and classical RWs. Consider a biased simple RW on $\mathbb{Z}$. That is to say, the probability to move one-step to the right is greater than that of moving to the left. In this case, the RW is obviously transient. At first glance, in the above model, when α^2 is greater than β^2, it looks something like a biased RW (P_1 moves to the left with probability α^2 and P_2 moves to the right with the same probability). However, if β^2 is nonzero, no matter how small, P_2 transitions to P_1 with probability β^2 and then it moves persistently to the left. This results in recurrent dynamics.

Remark 4.9. The recurrence notion used in this paper should be distinguished from monitored site recurrence, where one performs a measurement after each step and computes first-return probabilities for the monitored dynamics. The two notions are not equivalent in general, because the monitoring procedure modifies the evolution itself. Nevertheless, for the one-dimensional homogeneous examples considered above, our conclusions are consistent with known monitored site-recurrence criteria for OQRWs on the line; see, for instance, [8,19]. In this sense, the present potential-theoretic approach provides a complementary, non-measuring description of the same long-time behavior in these examples.

4.4.2. OQRWs on Cayley trees

Now, we consider little bit extended example of OQRWs on the semi-infinite Cayley trees. First we discuss a simple model of transient OQRW. Let $\Gamma_+^2=(L,E)$ be a semi-infinite Cayley tree of order 2 with the root x^0 (i.e. each vertex of Γ_+^2 has exactly 3 edges, except for the root x^0, which has 2 edges). Recall a coordinate structure in Γ_+^2: every vertex x (except for x^0) of Γ_+^2 has coordinates $(i_1,\dots,i_n)$, here $i_m\in\{1,2\}$, $1\le m\le n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1,\dots,i_n)$ form level n of the lattice which is denoted by W_n (see Fig. 1).

Let us consider a model OQRW on Γ_+^2. Assume that $B,C\in B(\mathcal{H})$ such that

$$B^*B+C^*C=\mathbb{1}$$

and there are two orthogonal projections P and Q with

$$B^*PB=P,\quad B^*QB=0,\quad C^*PC=0,\quad C^*QC=Q. \tag{4.11}$$

Now, we define B_j^i as follows:

$$B_j^i=\begin{cases}B, & i=(j,1),\\ C, & i=(j,2), \qquad j\in\Gamma_+^2.\\ 0, & \text{otherwise.}\end{cases}$$

Take $p_0 = (P+Q) \otimes |0\rangle\langle 0|$. Then,

$$\mathcal{T}^n(p_0) = P \otimes |(0\underbrace{11\dots1}_{n})\rangle\langle(0\underbrace{11\dots1}_{n})| + Q \otimes |(0\underbrace{22\dots2}_{n})\rangle\langle(0\underbrace{22\dots2}_{n})|.$$

Hence, for any $u = \sum_{j\in\Gamma_+^2} u_j \otimes |j\rangle$, one gets

$$\begin{aligned}\sum_{n\geq 0}\langle u, \mathcal{T}^n(p_0)u\rangle &= \|u_0\|^2 + \sum_{n=1}^{\infty}\left(\|Pu_{(0\underbrace{11\dots1}_{n})}\|^2 + \|Qu_{(0\underbrace{22\dots2}_{n})}\|^2\right)\\ &\leq \|u\|^2\end{aligned}$$

which implies the projection p_0 is transient.

The second model is an OQRW on a regular tree. Let $\mathbb{T}^3$ be a regular tree of order 3. Fix a root $x^0 \in \mathbb{T}^3$ and as before denote by W_n the set of sites in nth level, or the nth generation. Let $U, V, W \in B(\mathcal{H})$ be unitary operators and choose an $0 \leq r \leq 1/2$. We define B_j^i as follows:

$$B_0^i = \begin{cases} \frac{1}{\sqrt{3}}V, & i=(0,1),\\ \frac{1}{\sqrt{3}}U, & i=(0,2),\\ \frac{1}{\sqrt{3}}W, & i=(0,3),\\ 0, & \text{otherwise,}\end{cases} \qquad B_j^i = \begin{cases} \sqrt{r}\,V, & i=(j,1),\\ \sqrt{r}\,U, & i=(j,2),\\ \sqrt{1-2r}\,W, & j=(i,1),\ (i,2),\ \text{or } (i,3),\\ 0, & \text{otherwise.}\end{cases} \qquad j\in W_n,\ n\geq 1. \tag{4.12}$$

From the unitarity of the operators U, V, and W, the dynamics leaves the subalgebra generated by the projections in the set $\{\mathbb{1}\otimes|j\rangle\langle j| : j \in \mathbb{T}^3\}$ invariant and the action becomes simple. In fact, let $p_0 = \mathbb{1}\otimes|0\rangle\langle 0|$ be a projection supported at the origin. The one-step evolution is

$$\mathcal{T}(p_0) = (1-2r)\sum_{j\in W_1}\mathbb{1}\otimes|j\rangle\langle j|.$$

For $n \geq 1$, the n-step evolution becomes

$$\mathcal{T}^n(p_0) = \sum_{k=0}^{\infty} a_n(k)\sum_{j\in W_k}\mathbb{1}\otimes|j\rangle\langle j|, \tag{4.13}$$

where the coefficients satisfy the following recurrence relation:

$$\begin{aligned} a_{n+1}(k) &= 2ra_n(k+1) + (1-2r)a_n(k-1), \quad k\geq 1,\\ a_{n+1}(0) &= a_n(1).\end{aligned} \tag{4.14}$$

Let $P = (P(i,j))_{i,j\in\mathbb{N}_0}$ be a stochastic matrix on the nonnegative integer space $\mathbb{N}_0 = \{0,1,2,\cdots\}$ defined by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots\\ 1-2r & 0 & 2r & 0 & \cdots\\ 0 & 1-2r & 0 & 2r & \cdots\\ 0 & 0 & 1-2r & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4.15}$$

Considering $a_n = (a_n(k))_{k\in\mathbb{N}_0}$ as a column vector, we observe that the solution of the equation (4.14) is given by

$$a_n = P^n a_0, \quad a_0 = (1, 0, 0, \cdots)^T. \tag{4.16}$$

It is well-known that the classical Markov chain on $\mathbb{N}_0$ with a transition matrix P in (4.15) is transient if and only if $0 \le 1 - 2r < 2r \le 1$, or $1/4 < r \le 1/2$ [20, page 108]. With these preparations, we can state the following

Proposition 4.10. *The OQRW on the regular tree $\mathbb{T}^3$ with generating operators B_j^i given by (4.12) is transient if and only if $1/4 < r \le 1/2$. For $0 \le r \le 1/4$, all the projections $p_j = \mathbb{1} \otimes |j\rangle\langle j|$, $j \in \mathbb{T}^3$, are recurrent.*

Proof. First we investigate the dynamics of $p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$. Let $u = u_0 \otimes |0\rangle$ be any vector in the range of p_0. Then, by (4.13) it follows that

$$\sum_{n\ge 0} \langle u, \mathcal{T}^n(p_0)u\rangle = \|u\|^2 \sum_{n\ge 0} a_n(0). \tag{4.17}$$

Now defining $A(k) := \sum_{n=0}^{\infty} a_n(k)$, we sum both sides of (4.14) over $n \ge 0$ to get

$$A(k) - a_1(k) = 2rA(k+1) + (1-2r)A(k-1), \tag{4.18}$$

$$A(0) - a_0(0) = A(1) \tag{4.19}$$

Noticing that $a_1(k) = (1-2r)\delta_1(k)$, and defining $B(k) := A(k) - A(k-1)$, $k \ge 1$, the recurrence equation (4.18) can be rewritten as

$$\begin{aligned} B(k+1) &= \left(\frac{1-2r}{2r}\right) B(k), \quad k \ge 2, \\ B(2) &= \left(\frac{1-2r}{2r}\right)(B(1) - 1) = -\frac{1-2r}{r}, \end{aligned} \tag{4.20}$$

where we have used $B(1) = A(1) - A(0) = -a_0(0) = -1$ from (4.19). Therefore, (4.20) is solved as

$$B(1) = -1,\ B(2) = -\frac{1-2r}{r},\ B(k) = \begin{cases} -\left(\frac{1-2r}{r}\right)\left(\frac{1-2r}{2r}\right)^{k-2}, & 1-2r < 2r, \\ -\left(\frac{1-2r}{r}\right), & 1-2r = 2r, \quad k \ge 2. \\ -\left(\frac{1-2r}{r}\right)\left(\frac{1-2r}{2r}\right)^{k-2} \underset{k\to\infty}{\longrightarrow} -\infty, & 1-2r > 2r, \end{cases} \tag{4.21}$$

Plugging into the equation $A(k) = A(k+1) - B(k+1)$, $k \ge 0$, we conclude that $A(k)$ is finite if and only if $1 - 2r < 2r$ and in this case it is given by

$$A(k) = \frac{2r}{4r-1}\left(\frac{1-2r}{2r}\right)^k, \quad k \ge 0. \tag{4.22}$$

Therefore, when $u = u_0 \otimes |0\rangle\langle 0| \neq 0$, the right hand side of (4.17) is finite if and only if $1 - 2r < 2r$, or the projection p_0 is recurrent if and only if $0 \le 2r \le 1 - 2r \le 1$, i.e., $0 \le r \le 1/4$. In a quite similar fashion, we can prove that the projections $p_j = \mathbb{1} \otimes |j\rangle\langle j|$ are recurrent if and only if $0 \le r \le 1/4$. Finally, we show that the OQRW is transient if and only if $1/4 < r \le 1/2$. For this purpose, however, we have to rely on Definition 3.8. For each $m \in \mathbb{N}_0$, let

$$p_{m]} := \sum_{k=0}^{m} \sum_{j\in W_k} \mathbb{1} \otimes |j\rangle\langle j|$$

be the projection to the local system up to the mth level. Let $u=\sum_j u_j\otimes|j\rangle\langle j|\in\mathcal{H}\otimes\mathcal{K}$ be any vector. For $m=0$, in the transience regime $1/4<r\le 1/2$ and only for these values, we see from (4.13) and (4.22) that

$$\begin{aligned}\sum_{n=0}^{\infty}\langle u,\mathcal{T}^n(p_0)u\rangle &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_n(k)\sum_{j\in W_k}\|u_j\|^2\\ &= \left(\frac{2r}{4r-1}\right)\sum_{k=0}^{\infty}\left(\frac{1-2r}{2r}\right)^k\sum_{j\in W_k}\|u_j\|^2\\ &\le \left(\frac{2r}{4r-1}\right)\|u\|^2.\end{aligned}$$

Therefore, the operator $\mathcal{U}(p_0)$ is a bounded operator. Notice that by linearity, $\mathcal{T}^n(p_{m]})=\sum_{k=0}^{m}\times\sum_{j\in W_k}\mathcal{T}^n(p_j)$. We omit the details, but following a similar procedure as above, we can show that each $\mathcal{U}(p_j)$ is a bounded operator. Thus $\mathcal{U}(p_{m]})$ is also bounded as well. Since $\bigvee_{m\ge 0}p_{m]}=\mathbb{1}$, the criterion is proved. □

5. UQRWs

In this section we study the UQRWs.

5.1. *Definition of UQRWs*

We briefly introduce the 1-dimensional UQRWs. For the details we refer to the references [4,14,21,23,24]. In [22], UQRWs were studied from the viewpoint of quantum Markov chains. In [18], Gudder discussed UQRWs from the viewpoint of TOMs, but we adopt the method in [22]. Let us review it, adjusting the notation to the present setting. We take $\Lambda=\mathbb{Z}$, so $\mathcal{K}=l^2(\Lambda)$ and let $\mathcal{H}=\mathbb{C}^2$, the latter is called the coin Hilbert space. A canonical basis of $\mathcal{H}$ is denoted by $|L\rangle,|R\rangle$, which can be thought of as the two sides of a ‘quantum coin’: after the coin is flipped, if $|L\rangle$ appears, the particle moves left one unit, and if $|R\rangle$ appears, the particle moves right one unit. The quantum coin is given by a 2×2 unitary matrix C and we decompose it into two parts:

$$C=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ c & d\end{bmatrix}=:P+Q \tag{5.1}$$

Define a translation T on $\mathcal{H}\otimes\mathcal{K}$ by

$$T(\sum_j\psi_j\otimes|j\rangle)=\sum_j\psi_{j+1}\otimes|j\rangle.$$

Now define an operator U on $\mathcal{H}\otimes\mathcal{K}$ by

$$U:=P\otimes T+Q\otimes T^*.$$

It is easy to see that U is a unitary operator. We regard any unit vector $\psi=\sum_j\psi_j\otimes|j\rangle$ also as a pure state $|\psi\rangle\langle\psi|$ on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. The 1-dimensional UQRW is an evolution on the space of pure states which can now be stated in the following way.

Definition 5.1. Given a unitary coin matrix C, the **unitary quantum random walk (UQRW)** is defined by

$$\mathcal{M}(\psi)\equiv\mathcal{M}(|\psi\rangle\langle\psi|):=U|\psi\rangle\langle\psi|U^*,\quad \|\psi^2\|=1. \tag{5.2}$$

Remark 5.2. Obviously the UQRW can be extended to $\mathcal{I}(\mathcal{H}\otimes\mathcal{K})$, the space of trace class operators on $\mathcal{H}\otimes\mathcal{K}$. In this sense, one can say that the UQRW is a TOM on $\mathcal{I}(\mathcal{H}\otimes\mathcal{K})$.

As in OQRWs, we define a dual process $\mathcal{T}$ on $\mathcal{A}$ by

$$\mathcal{T}(x) := U^* x U, \quad x \in \mathcal{A}. \tag{5.3}$$

For each $i\in\mathbb{Z}$, let p_i be the projection on $\mathcal{H}\otimes\mathcal{K}$ defined by

$$p_i\Big(\sum_j \psi_j\otimes|j\rangle\Big)=\psi_i\otimes|i\rangle.$$

Lemma 5.3. *For any unit vector $\psi=\sum_j\psi_j\otimes|j\rangle\in\mathcal{H}\otimes\mathcal{K}$, we have*

$$\mathrm{Tr}\,\big(p_i\mathcal{M}(\psi)\big)=\|P\psi_{i+1}+Q\psi_{i-1}\|^2. \tag{5.4}$$

Particularly, the value gives the probability of finding the particle at site i after one step movement when it started from the state ψ.

Proof. We have

$$\mathrm{Tr}\,\big(p_i\mathcal{M}(\psi)\big)=\mathrm{Tr}(p_iU|\psi\rangle\langle\psi|U^*)=\|p_iU\psi\|^2.$$

But by definition of U we have $p_iU\psi=P\psi_{i+1}+Q\psi_{i-1}$. □

The following proposition gives the probabilistic feature of UQRW. Let X_n denote the random variable of the position of the UQRW at time n.

Proposition 5.4. *Starting from an initial state $\psi^{(0)}=\sum_j\psi_j^{(0)}\otimes|j\rangle$, after nth run of the UQRW, the probability to find the particle at site i is given by*

$$\mathbb{P}(X_n=i)=\mathrm{Tr}\,\big(p_i\mathcal{M}^n(\psi^{(0)})\big)=\langle\psi^{(0)},\mathcal{T}^n(p_i)\psi^{(0)}\rangle.$$

Proof. The first equality follows from the definition of UQRW and Lemma 5.3. The second equality follows from the dual relation:

$$\begin{aligned}\mathrm{Tr}\,\big(p_i\mathcal{M}^n(\psi^{(0)})\big)&=\mathrm{Tr}\,\big(p_iU^n|\psi^{(0)}\rangle\langle\psi^{(0)}|U^{*n}\big)\\&=\mathrm{Tr}\,\big(|\psi^{(0)}\rangle\langle\psi^{(0)}|U^{*n}p_iU^n\big)\\&=\mathrm{Tr}\,\big(|\psi^{(0)}\rangle\langle\psi^{(0)}|\mathcal{T}^n(p_i)\big)\\&=\langle\psi^{(0)},\mathcal{T}^n(p_i)\psi^{(0)}\rangle.\quad\square\end{aligned}$$

5.2. Potential theory for UQRWs

Now we use the quantum Markov operator $\mathcal{T}$ defined in (5.3) to study the potential theory for UQRWs. First we discuss a very simple sufficient condition for the reducibility.

Theorem 5.5. *If there is a nontrivial projection $h\in\mathcal{A}$ such that $[U,h]=0$, then the quantum Markov operator $\mathcal{T}$ for the UQRW is reducible. In this case h is $\mathcal{T}$-harmonic. In particular, if h_0 is a projection in $\mathbb{C}^2$ such that $[P,h_0]=0$ and $[Q,h_0]=0$, then the projection $h=h_0\otimes\mathbb{1}$ in $\mathcal{H}\otimes\mathcal{K}$ satisfies $[U,h]=0$.*

Proof. If $[U,h]=0$, then it is also true that $[U^*,h]=0$. Therefore, $\mathcal{T}(h)=U^*hU=h$, meaning that h is $\mathcal{T}$-harmonic. The second part is trivial. □

Example 5.6. Let $C=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $|a|=|d|=1$ and P_1, P_2 the projections defined in (4.8). The action of U on the canonical basis is particularly simple:

$$U(|L\rangle \otimes |j\rangle) = a\,|L\rangle \otimes |j-1\rangle, \qquad U(|R\rangle \otimes |j\rangle) = d\,|R\rangle \otimes |j+1\rangle.$$

Hence the two closed subspaces

$$\mathcal{H}_L := P_1\mathcal{H} \otimes l^2(\mathbb{Z}), \qquad \mathcal{H}_R := P_2\mathcal{H} \otimes l^2(\mathbb{Z})$$

are invariant under U. Let $h_0 = P_1$ (or P_2). Then, obviously $[P,h_0]=[Q,h_0]=0$. Thus, $h=h_0\otimes \mathbb{1}$ is $\mathcal{T}$-harmonic and $\mathcal{T}$ is reducible. In other words, the walk splits into two invariant chiral sectors, making reducibility completely transparent.

The following three examples illustrate three distinct behaviors of the UQRW framework: reducibility through invariant chiral sectors, ballistic transience, and genuine periodic recurrence. Let us also note that in the unitary setting the monitored recurrence formalism is based on repeated projections onto the complement of a given site or subspace, and therefore it does not coincide a priori with the potential-theoretic notion used here. Nevertheless, in the concrete examples below the conclusions of the two approaches can be compared directly.

Example 5.7. Let us consider again $C=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $|a|=|d|=1$. We will see that the corresponding quantum Markov operator $\mathcal{T}$ is transient. For each $n\in\mathbb{N}$, let

$$p_{[-n,n]} := \sum_{j=-n}^{n} \mathbb{1}\otimes |j\rangle\langle j| \in \mathcal{A}$$

be a projection. It is easy to compute that

$$\mathcal{T}^k(p_{[-n,n]}) = \sum_{j=-n}^{n} \big(P_1 \otimes |j+k\rangle\langle j+k| + P_2 \otimes |j-k\rangle\langle j-k|\big).$$

Now fix a basis vector $|L\rangle\otimes|m\rangle$. It belongs to the support of $\mathcal{T}^k(p_{[-n,n]})$ if and only if $m=j+k$ for some $j\in[-n,n]$, i.e. if and only if $m-k\in[-n,n]$. Hence this can happen for at most $2n+1$ values of $k\geq 0$. The same argument applies to $|R\rangle\otimes|m\rangle$. Therefore,

$$\sum_{k\geq 0} \mathcal{T}^k(p_{[-n,n]}) \leq (2n+1)\mathbb{1},$$

which shows that $\mathcal{U}(p_{[-n,n]})$ is bounded. Since $\bigvee_{n\geq 0} p_{[-n,n]} = \mathbb{1}$, Definition 3.8 implies that $\mathcal{T}$ is transient. In the monitored formalism, the same conclusion is immediate at the level of sites: once the walker leaves the origin, each chiral component propagates ballistically in one direction and never returns.

Next we provide with an example of recurrence.

Example 5.8. Let $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and one computes

$$U(|L\rangle \otimes |j\rangle) = |R\rangle \otimes |j+1\rangle, \qquad U(|R\rangle \otimes |j\rangle) = |L\rangle \otimes |j-1\rangle.$$

Hence $U^2 = \mathbb{1}$, so the dynamics is periodic with period 2. In particular, for every site projection $p_j = \mathbb{1} \otimes |j\rangle\langle j|$,

$$\mathcal{T}^{2n}(p_j) = p_j, \qquad n \geq 0,$$

which implies that every p_j is recurrent. Now let P_1 and P_2 denote again the canonical projections on $\mathbb{C}^2$, let $h_0 = P_1$ (or P_2), and define $h = h_0 \otimes \mathbb{1}$. One computes that

$$U^* P_1 U = P_2 \otimes \mathbb{1} \text{ and } U^* P_2 U = P_1 \otimes \mathbb{1}.$$

Therefore,

$$\mathcal{T}^{2n}(h) = h \text{ and } \mathcal{T}^{2n-1}(h) = h^{\perp}.$$

If $u \in \mathrm{Ran}(h) \setminus \{0\}$, then

$$\sum_{n \geq 0} \langle u, \mathcal{T}^n(h) u\rangle = \sum_{n \geq 0} \langle u, \mathcal{T}^{2n}(h) u\rangle = \sum_{n \geq 0} \|u\|^2 = \infty.$$

We conclude that h is $\mathcal{T}$-recurrent. The monitored interpretation agrees with this conclusion: a walker started at a fixed site returns to that site after exactly two steps.

Declaration of competing interest

There is no conflict of interest in this article.

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Appendix A. Proof of Proposition 4.7

Equation (4.10) says that the vector space generated by the projections P_1 and P_2 are invariant under the action of $\mathcal{E}_j^{i*}$'s. So, starting from $x^{(0)} = p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$ we can write

$$x^{(n)} \equiv \mathcal{T}^n(x^{(0)}) = \sum_{j \in \mathbb{Z}} \big(a_n(j) P_1 + b_n(j) P_2\big) \otimes |j\rangle\langle j|.$$

By the rule (4.10), the sequences $(a_n(j))$ and $(b_n(j))$ satisfy the recursion relation:

$$\begin{bmatrix} a_{n+1}(j) \\ b_{n+1}(j) \end{bmatrix} = \begin{bmatrix} \alpha^2 a_n(j-1) + \beta^2 b_n(j-1) \\ \beta^2 a_n(j+1) + \alpha^2 b_n(j+1) \end{bmatrix}, \quad n \geq 0,\ j \in \mathbb{Z}. \tag{A.1}$$

Now defining unit vectors $e_j(k) := \frac{1}{\sqrt{2\pi}} e^{ijk}$, $j \in \mathbb{Z}$, $k \in [-\pi, \pi]$, the set $\{e_j : j \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2([-\pi, \pi])$. For a vector $x = \oplus_{j\in\mathbb{Z}} \begin{bmatrix} a(j) \\ b(j) \end{bmatrix} \equiv \left(\begin{bmatrix} a(j) \\ b(j) \end{bmatrix} \right)_{j\in\mathbb{Z}} \in \oplus_{j\in\mathbb{Z}} \mathbb{C}^2 \cong l^2(\mathbb{Z}, \mathbb{C}^2)$, we define its Fourier transform by

$$\widehat{x}(k) := \begin{bmatrix} \sum_{j\in\mathbb{Z}} a(j) e_j(k) \\ \sum_{j\in\mathbb{Z}} b(j) e_j(k) \end{bmatrix} \in L^2([-\pi, \pi], \mathbb{C}^2).$$

Its inverse Fourier transform is given by

$$\check{\widehat{x}}(j) = \int_{-\pi}^{\pi} e_{-j}(k) \widehat{x}(k) dk.$$

Then, the recursion relation (A.1) shows that

$$\widehat{x^{(n)}}(k) = P(k)^n \widehat{x^{(0)}}(k), \tag{A.2}$$

where $P(k)$ is a 2×2 matrix given by

$$P(k) = \begin{bmatrix} e^{ik}\alpha^2 & e^{ik}\beta^2 \\ e^{-ik}\beta^2 & e^{-ik}\alpha^2 \end{bmatrix}. \tag{A.3}$$

To see the asymptotic behavior of $P(k)^n$ as $n \to \infty$, we need to investigate the eigenvalues of $P(k)$. Since the cases $\alpha = 1$ and $\alpha = 0$ have been already studied, we assume $0 < \alpha^2 < 1$. The matrix $P(k)$ has eigenvalues

$$\lambda_\pm(k) = \alpha^2 \cos k \pm \sqrt{\alpha^4 \cos^2 k - (2\alpha^2 - 1)}. \tag{A.4}$$

Case 1: $0 < \alpha^2 \leq 1/2$.
In this case the quantity inside the square root symbol is always positive. Therefore, $P(k)$ has two real eigenvalues. As a function of k, the eigenvalue $\lambda_+(k)$ has value 1 at $k = 0$ and is strictly less than 1 and greater than 0 at other values of k. Similarly, the function $\lambda_-(k)$ multiplied by -1 has its value 1 at $k = \pi$ $(= -\pi)$ and lies between 0 and 1 elsewhere. So, the asymptotic behavior of $P(k)^n$ is governed by the values near $k = 0$. Therefore, when $k \approx 0$ we can write the value of $\lambda_+(k)$ in the following way:

$$\begin{aligned}
\lambda_+(k) &= \alpha^2 \cos k + \sqrt{\alpha^4 \cos^2 k - (2\alpha^2 - 1)} \\
&= \alpha^2 \sqrt{1 - \sin^2 k} + \sqrt{(1 - \alpha^2)^2 - \alpha^4 \sin^2 k} \\
&\approx \alpha^2 \left(1 - \frac{1}{2} \sin^2 k\right) + (1 - \alpha^2) \left(1 - \frac{1}{2} \left(\frac{\alpha^2}{1 - \alpha^2}\right)^2 \sin^2 k\right) \\
&= 1 - \frac{\alpha^2}{2(1 - \alpha^2)} \sin^2 k \\
&\geq 1 - \frac{\alpha^2}{2(1 - \alpha^2)} k^2
\end{aligned}$$

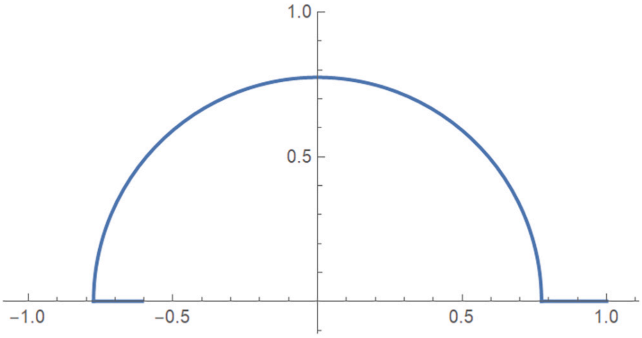

Fig. 2. The graph of $\lambda_+(k)$ $(\alpha^2 = 0.8)$.

Therefore, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \lambda_+(k)^n dk &\geq \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \left(1 - \frac{\alpha^2}{2(1-\alpha^2)} k^2\right)^n dk \\ &\geq \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \left(1 - \frac{\alpha^2}{2(1-\alpha^2)} \frac{1}{n}\right)^n dk \\ &\approx e^{-\alpha^2/(2(1-\alpha^2))} \frac{2}{\sqrt{n}}.\end{aligned}$$

The estimation of $(-\lambda_-(k))^n$ can be done similarly with a same result (k being changed to $k - \pi$ and we are considering the behavior near $k = \pi$). Now, since the inverse transform of $\widehat{x^{(n)}}(k)$ at $j = 0$ is a constant multiple of the left-hand side of the above equation, summing over n leads to divergence. It proves that the projection $p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$ is recurrent. Furthermore, by the same argument, we can say that any projection $p_j = \mathbb{1} \otimes |j\rangle\langle j|$ is also recurrent.

Case 2: $1/2 < \alpha^2 < 1$.
In this case the eigenvalue $\lambda_+(k)$ can be complex valued for certain values of k (it is the same for $\lambda_-(k)$). Nonetheless, the behavior of the function is rather simple. Since the quantity inside the square root symbol can be written as $(1-\alpha^2)^2 - \alpha^4 \sin^2 k$, the value of $\lambda_+(k)$ has its value 1 at $k = 0$ and as k increases, it remains positive up to $k = \sin^{-1}\left(\frac{1-\alpha^2}{\alpha^2}\right)$, and it runs over a circle of radius $\sqrt{2\alpha^2 - 1}$ on the complex plane, stays on the real line, and returns back to the point 1 (see Fig. 2). Therefore, here again, the asymptotic behavior of $\lambda_+(k)^n$ as $n \to \infty$ is governed by its values near $k = 0$. Now, exactly as in Case 1, by estimating the values of $\lambda_+(k)^n$ around $k = 0$, we obtain $\int_{-\pi}^{\pi} \lambda_+(k)^n dk = O(1/\sqrt{n})$ showing that the projection $p_0 = \mathbb{1} \otimes |0\rangle\langle 0|$ as well as all other projections $p_j = \mathbb{1} \otimes |j\rangle\langle j|$ are recurrent. This completes the proof. □

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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