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INCREASING LIMIT OF REPRODUCING KERNELS

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ABSTRACT. We discuss the limit of increasing reproducing kernels and construct the corresponding reproducing kernel Hilbert space. Given an increasing sequence of reproducing kernel Hilbert spaces with their norms decreasing, we can find a limit of the sequence of those reproducing kernels. Thereby we can construct a reproducing kernel Hilbert space on a subset of common underlying sets. The proof corrects an error in that of Aronszajn. We discuss the error by giving an example.

1. Introduction. The theory of reproducing kernels is interesting in itself and has many applications in several areas [**1, 2**]. Aronszajn, in his survey paper [**1**], introduced the theory of reproducing kernel Hilbert spaces quite extensively. From that paper one can learn not only the basic definitions and properties but also many construction methods from given kernels, for instance, restriction theory and limit theories. There are also many concrete examples in [**1**]. In practice, the limit theories for reproducing kernels are very useful and also very important when one wants to construct a new space from a given sequence of spaces. In this paper we focus on the limit theory of increasing reproducing kernels.

As for a motivation, the present author has recently developed a dual relation in a dual pair of reproducing kernel Hilbert spaces (rigged spaces) and applied it to show the Gibbsianness of certain determinantal point processes [**4**] (see also [**3**]). There, the limit theory of increasing reproducing kernels played an important role.

As mentioned above the theory of restrictions and limits of reproducing kernels was well established by Aronszajn [**1**]. But the proof for the construction of an increasing limit of a sequence of reproducing

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kernels in [**1**] contains an error. The purpose of this paper is to correct it. Thereby we provide a proof for the result. In Section 3 we give an example which shows why Aronszajn's method needs a correction.

2. Preliminaries and main result. In this section, for the readers' convenience, we briefly recall the definition of reproducing kernel Hilbert spaces, and then state the main result. Most of the following contents come from [**1**].

Let E be any set and F a class of functions on E forming a Hilbert space. F may be a complex or real Hilbert space, but we only consider the complex case. By complexifying it, if necessary, the real case can be dealt with equally well. Let $(\cdot,\cdot)$ denote the inner product in F (linear in the second variable). A function $K(x,y)$, $x,y \in E$, is called a reproducing kernel (RK in short) of F if

(i) For every $y \in E$, the function $K(\cdot,y)$ belongs to F,

(ii) The reproducing property: for every $y \in E$ and $f \in \mathsf{F}$, $f(y) = (K(\cdot,y), f)$.

The Hilbert space F possessing an RK is called a reproducing kernel Hilbert space (RKHS shortly). The basic property of any RK $K(x,y)$ is that it is a positive definite function in the following sense: for any finite complex numbers $z_1,\dots,z_n \in \mathbf{C}$,

$$\sum_{i,j=1}^{n} \overline{z_i} K(x_i,x_j) z_j \geq 0, \tag{2.1}$$

for all $x_1,\dots,x_n \in E$. The converse is also true: if $K(x,y)$ is a positive definite function on $E \times E$, there exists a unique RKHS with RK $K(x,y)$.

The main subject we want to discuss in this paper is the limit of increasing sequence of reproducing kernels. It is explained in [**1**, Sec. 9, Part I]. The following settings are the same as in [**1**].

Let $\{E_n\}$ be a decreasing sequence of sets, E their intersection:

$$E = \bigcap_{n=1}^{\infty} E_n, \quad E_1 \supset E_2 \supset \cdots . \tag{2.2}$$

For each n, let F_n be a class of functions defined in E_n. For $m \geq n$, we define the restriction f_{nm} of $f_n \in \mathsf{F}_n$ to the set E_m. We suppose that

F_n form an increasing sequence:

$$\text{for every } f_n \in \mathsf{F}_n \text{ and every } m \geq n, \quad f_{nm} \in \mathsf{F}_m. \tag{2.3}$$

We suppose further that the norm $\|\cdot\|_n$ defined in F_n form a decreasing sequence:

$$\text{for every } f_n \in \mathsf{F}_n \text{ and every } m \geq n, \quad \|f_{nm}\|_m \leq \|f_n\|_n. \tag{2.4}$$

Finally, we suppose that every F_n possesses an RK $K_n(x,y)$. As was shown in [**1**], we have

$$K_{nm} \ll K_m \quad \text{for } m \geq n, \tag{2.5}$$

meaning that $K_m(x,y) - K_{nm}(x,y)$ is a positive definite function on E_m. Therefore, for each $y \in E$, $\{K_m(y,y)\}$ is an increasing sequence of positive numbers. Its limit may be infinite. We define, consequently,

$$E_0 := \{y \in E : \, K_0(y,y) := \lim_{m\to\infty} K_m(y,y) < \infty\}. \tag{2.6}$$

We suppose that E_0 is not empty. Let F_0 be the class of all restrictions f_{n0} of functions $f_n \in \mathsf{F}_n$ $(n = 1, 2, \dots)$ to the set E_0. From (2.4) the sequence $\{\|f_{nk}\|_k\}_{k\geq n}$ is decreasing and we define for any element $f \in \mathsf{F}_0$ (i.e., $f = f_{n0}$ for some $f_n \in \mathsf{F}_n$)

$$\|f\|_0 := \inf \lim_{k\to\infty} \|f_{nk}\|_k, \tag{2.7}$$

where the infimum is taken over all functions $f_n \in \mathsf{F}_n$, $n \geq 1$, whose restriction to E_0 is f.

Proposition 2.1. *The function $\|f\|_0^2$ in (2.7) is a quadratic form and $\|\cdot\|_0$ is a norm on F_0.*

Remark 2.2. In [**1**], the norm $\|\cdot\|_0$ was erroneously defined as

$$\|f_{n0}\|_0 = \lim_{k\to\infty} \|f_{nk}\|_k. \tag{2.8}$$

But from this alone we cannot guarantee the well-definedness: we cannot confirm $\|f_{n0}\|_0 = \|g_{n0}\|_0$ for different f_n and g_n in F_n with

$f_{n0} = g_{n0}$. That is, it might be the case $\|0\|_0 > 0$ for the zero vector in F_0, and hence $\|\cdot\|_0$ is not a norm. In the next section we will show that this situation really occurs in the examples. Since the statement and the proof of Aronszajn in [**1**, Theorem II, page 367] presumes that $\|\cdot\|_0$ is a norm in F_0, it is necessary to correct it.

After getting a suitable norm $\|\cdot\|_0$ for F_0, we can now state the result properly. The following was stated in [**1**, Theorem II, page 367].

Theorem 2.3. *For each $y \in E_0$, the restrictions $\{K_{n0}(\cdot, y)\}$ form a Cauchy sequence in F_0. Moreover, for each $x \in E_0$, the sequence $\{K_{n0}(x, y)\}$ converges to a number $K^*(x, y)$ which is an* RK *for an* RKHS F_0^* *on E_0.*

The way of the proof for the main result is the same as in [**1**]. We only need to check that the new norm replaces well the one in Aronszajn's arguments. In order to make the paper complete, however, we present it in the Appendix.

3. A counter example. In this section we show why it is needed to take the infimum in the definition of the norm in (2.7). What we will show is that if the "inf" in the definition of $\|\cdot\|_0$ in (2.7) were missing, $\|\cdot\|_0$ is not a norm in general because the zero vector may have nonzero norm.

Let $E := \mathbf{N}$ and $\{e_n\}_{n\geq 1}$ be the usual basis of the Hilbert space $\mathcal{H} := l^2(E)$. Let $A := B^*B$, where B is a bounded linear operator on $\mathcal{H}$ defined by

$$Be_n = \begin{cases} e_1 & n = 1, \\ \frac{1}{n}(e_1 + e_n) & n \geq 2, \end{cases} \tag{3.1}$$

and by a linear extension. For each $\varepsilon > 0$ we let

$$A(\varepsilon) := A + \varepsilon. \tag{3.2}$$

Notice that for each $\varepsilon > 0$, $A(\varepsilon) \geq \varepsilon > 0$ and so it is invertible; we let $B(\varepsilon)$ be its inverse:

$$B(\varepsilon) := A(\varepsilon)^{-1}. \tag{3.3}$$

For $x, y \in E$, let $B(\varepsilon)(x, y)$ be the representation of $B(\varepsilon)$ with respect to the basis $\{e_n\}_{n\geq 1}$:

$$B(\varepsilon)(x,y) := (e_x, B(\varepsilon)e_y), \quad x, y \in E \equiv \mathbf{N}, \tag{3.4}$$

where $(\cdot,\cdot)$ denotes the inner product of $\mathcal{H}$ and the corresponding norm will be denoted by $\|\cdot\|$. For every $\varepsilon > 0$, let F_ε be the RKHS with RK $B(\varepsilon)(x, y)$. Notice that F_ε consists of the vectors of $\mathcal{H}$ with a new inner product:

$$(f, g)_\varepsilon := (f, A(\varepsilon)g), \quad f, g \in \mathcal{H}. \tag{3.5}$$

In fact, since $A(\varepsilon)$ and $B(\varepsilon)$ are bounded and self-adjoint operators on $\mathcal{H}$, $B(\varepsilon)(\cdot, y) = B(\varepsilon)e_y \in \mathsf{F}_\varepsilon \equiv \mathcal{H}$ and for every $f \in \mathsf{F}_\varepsilon$ and $y \in E$,

$$\begin{aligned}(B(\varepsilon)(\cdot,y), f)_\varepsilon &= (B(\varepsilon)e_y, A(\varepsilon)f) \\ &= (e_y, f) = f(y).\end{aligned} \tag{3.6}$$

By denoting $\|\cdot\|_\varepsilon$ the norm corresponding to the inner product $(\cdot,\cdot)_\varepsilon$, we see that $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent for all $\varepsilon > 0$. The system of RKHS's $\{\mathsf{F}_\varepsilon\}_{\varepsilon>0}$ will play the role of the sequence $\{\mathsf{F}_n\}$ of the main body. (If necessary, we just take a sequence $s_n \downarrow 0$ and consider $\{\mathsf{F}_{s_n}\}$; but, it is not harmful to consider the continuous parameter.) Since the space E_ε, on which F_ε is constructed, is E for all $\varepsilon > 0$, we have

$$\bigcap_{\varepsilon>0} E_\varepsilon = E.$$

Now we want to find E_0 that is defined by

$$E_0 = \Big\{ y \in E : \ K_0(y,y) := \lim_{\varepsilon\downarrow 0} B(\varepsilon)(y,y) < \infty \Big\}. \tag{3.7}$$

Lemma 3.1. $E_0 = E \setminus \{1\}$.

Proof. For a moment we denote $\|\cdot\|_\varepsilon$ by $\|\cdot\|_{\varepsilon,-}$ and define a new norm $\|\cdot\|_{\varepsilon,+}$ corresponding to the inner product

$$(f, g)_{\varepsilon,+} := (f, B(\varepsilon)g), \quad f, g \in \mathcal{H}. \tag{3.8}$$

To compute the components $B(\varepsilon)(y,y)$, we can apply the variational principle of [**4**] to the dual pair of RKHS's with RK's $A(\varepsilon)(x,y)$ and $B(\varepsilon)(x,y)$. We decompose E into three disjoint sets: $E=\{y\}\cup(E\setminus\{y\})\cup\varnothing$. Then by [**4**, Theorem 2.4] (see also [**3**, Theorem 6.3]) we get

$$B(\varepsilon)(y,y)=\Big[\lim_{\Lambda\uparrow E}\inf_{f_\Lambda\in\mathcal{H}_{\Lambda\cap(E\setminus\{y\})}}(e_y-f_\Lambda, A(\varepsilon)(e_y-f_\Lambda))\Big]^{-1}, \tag{3.9}$$

where $\Lambda\uparrow E$ means that Λ increases to E through finite sets and $\mathcal{H}_{\Lambda\cap(E\setminus\{y\})}$ is the subspace of $\mathcal{H}$ spanned by $\{e_x: x\in\Lambda\cap(E\setminus\{y\})\}$. Since $A(\varepsilon)$ is a bounded operator the quantity inside the bracket $[\cdots]$ in (3.9) is equal to

$$\alpha(\varepsilon):=\inf_{f\in\mathcal{H}_{E\setminus\{y\}}}(e_y-f, A(\varepsilon)(e_y-f)). \tag{3.10}$$

When $y=1$, from the definition of the operator $A(\varepsilon)$, we have for any $f\in\mathcal{H}_{E\setminus\{1\}}$,

$$\begin{aligned}(3.11)\quad &(e_1-f, A(\varepsilon)(e_1-f))\\ &=\|B(e_1-f)\|^2+\varepsilon\|e_1-f\|^2\\ &=\Big|1-\sum_{m\ge 2}\frac{1}{m}f_m\Big|^2+\sum_{m\ge2}\frac{1}{m^2}|f_m|^2+\varepsilon\Big(1+\sum_{m\ge2}|f_m|^2\Big),\end{aligned}$$

where $f=\sum_{m\ge2}f_me_m$. By varying the coefficients $\{f_m\}$ we find the infimum value of (3.11). For it, we first minimize the right hand side of (3.11) under the constraint (we may assume that f_m's are real)

$$\sum_{m\ge2}\frac{1}{m}f_m=a. \tag{3.12}$$

By the Lagrange multiplier method, the minimum is attained for f_m's that satisfy for all $m\ge 2$

$$2\frac{1}{m^2}f_m+2\varepsilon f_m=2\lambda\frac{1}{m}, \tag{3.13}$$

where λ is a constant. From (3.13) we get

$$f_m=\Big(\varepsilon+\frac{1}{m^2}\Big)^{-1}\frac{\lambda}{m},\quad m\ge 2. \tag{3.14}$$

Inserting these values into (3.12) we get

$$\lambda = b(\varepsilon)a; \tag{3.15}$$

here

$$b(\varepsilon) := \left[\sum_{m\geq 2} \frac{1}{m^2} \left(\varepsilon + \frac{1}{m^2} \right)^{-1} \right]^{-1}$$

and notice that $b(\varepsilon) \to 0$ as $\varepsilon \to 0$. The infimum of (3.11) under (3.12) is then

$$(1-a)^2 + b(\varepsilon)a^2 + \varepsilon. \tag{3.16}$$

Now the quadratic form has infimum value, which is $\alpha(\varepsilon)$ in (3.10),

$$\alpha(\varepsilon) = \frac{b(\varepsilon)}{1+b(\varepsilon)} + \varepsilon. \tag{3.17}$$

Since $b(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, and from (3.9) we conclude that

$$B(\varepsilon)(1,1) = \alpha(\varepsilon)^{-1} \to \infty \quad \text{as } \varepsilon \to 0. \tag{3.18}$$

For the points $y \equiv n \in E$ with $n \geq 2$, by similar arguments as above, we have

$$B(\varepsilon)(n,n) = \left[\frac{1}{n^2} \left(1 + \frac{b'(\varepsilon)}{1+b'(\varepsilon)} \right) + \varepsilon \right]^{-1}, \tag{3.19}$$

where

$$b'(\varepsilon) := \left[\frac{1}{\varepsilon} + \sum_{\substack{m\geq 1 \\ m\neq n}} \frac{1}{m^2(m^{-2}+\varepsilon)} \right]^{-1} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Therefore we get

$$\lim_{\varepsilon\to 0} B(\varepsilon)(n,n) = n^2 < \infty, \quad n \geq 2. \tag{3.20}$$

This completes the proof. □

Now choose the vector $f_\varepsilon \equiv e_1 = (1, 0, 0, \dots) \in \mathsf{F}_\varepsilon$. For $0 < \varepsilon' < \varepsilon$, since $E_{\varepsilon'} = E_\varepsilon = E$, the restriction $f_{\varepsilon\varepsilon'}$ of f_ε to $E_{\varepsilon'}$ is again f_ε. Thus

$$\begin{aligned}\lim_{\varepsilon'\to 0} \|f_{\varepsilon\varepsilon'}\|_{\varepsilon'}^2 &= \lim_{\varepsilon'\to 0} \|e_1\|_{\varepsilon'}^2 \\ &= \lim_{\varepsilon'\to 0} (e_1, A(\varepsilon')e_1) \\ &= \lim_{\varepsilon'\to 0} (\|Be_1\|^2 + \varepsilon') = 1 > 0.\end{aligned} \tag{3.21}$$

On the other hand the vector $f_{\varepsilon 0}$, the restriction of f_ε to E_0, is the zero vector. Therefore if the "inf" were missing in (2.7), we would get $\|0\|_0 > 0$, that is, the norm is not well-defined.

APPENDIX

A. Proof of Theorem 2.3. We start with a proof of Proposition 2.1.

Proof of Proposition 2.1. We first show that the function $\|\cdot\|_0$ is a norm. It obviously holds that $\|0\|_0 = 0$. Now suppose that $\|f\|_0 = 0$ for some vector $f \in \mathsf{F}_0$. For any $\varepsilon > 0$ there is a vector $f_n \in \mathsf{F}_n$ such that $f_{n0} = f$ and

$$\lim_{k\to\infty} \|f_{nk}\|_k < \varepsilon.$$

Then, for any $y \in E_0$,

$$\begin{aligned}|f(y)| &= |f_{n0}(y)| = |f_{nk}(y)| \\ &= |(K_k(\cdot, y), f_{nk})_k| \\ &\le \|f_{nk}\|_k K_k(y,y)^{1/2} \\ &\to \Big(\lim_{k\to\infty} \|f_{nk}\|_k\Big) K_0(y,y)^{1/2} \\ &< \varepsilon K_0(y,y)^{1/2}.\end{aligned} \tag{A.1}$$

Since $\varepsilon > 0$ is arbitrary $f(y) = 0$; hence, $f = 0$. In order to show that $\|\cdot\|_0^2$ is a quadratic form it is enough to show that $\|\cdot\|_0$ satisfies the parallelogram law:

$$\|f+g\|_0^2 + \|f-g\|_0^2 = 2\|f\|_0^2 + 2\|g\|_0^2.$$

Then $\|\cdot\|_0$ is the natural norm coming from an inner product. Given an $\varepsilon > 0$, suppose that u_n and v_n are the vectors in F_n for some $n \geq 1$ such that $u_{n0} = f$ and $v_{n0} = g$, and, moreover,

$$\lim_{k\to\infty} \|u_{nk}\|_k \leq \|f\|_0 + \varepsilon$$

and

$$\lim_{k\to\infty} \|v_{nk}\|_k \leq \|g\|_0 + \varepsilon.$$

Then since $u_{n0} + v_{n0} = f + g$ and $u_{n0} - v_{n0} = f - g$,

$$\begin{aligned}\|f+g\|_0^2 + \|f-g\|_0^2 &\leq \lim_{k\to\infty} \left(\|u_{nk}+v_{nk}\|_k^2 + \|u_{nk}-v_{nk}\|_k^2\right)\\ &= \lim_{k\to\infty} 2\left(\|u_{nk}\|_k^2 + \|v_{nk}\|_k^2\right)\\ &\leq 2(\|f\|_0+\varepsilon)^2 + 2(\|g\|_0+\varepsilon)^2,\end{aligned}$$

where the equality in the second line comes from the parallelogram law for the space F_k. Since ε is arbitrary we have

$$\|f+g\|_0^2 + \|f-g\|_0^2 \leq 2\|f\|_0^2 + 2\|g\|_0^2, \quad f, g \in \mathsf{F}_0. \tag{A.2}$$

In a similar way we can show the reverse inequality and this completes the proof. □

We will denote by $(\cdot,\cdot)_0$ the inner product for the quadratic form $\|\cdot\|_0^2$.

If one wants to consider RK's for a given class of functions on a set, the theory of restriction of RK's and the functional completion problem play the important roles. First we introduce the restriction theory of RK's.

Theorem A.1 ([**1**, page 351]). *If K is the* RK *of a class of functions defined on a set E with norm $\|\cdot\|$, then K restricted to a subset $E_1 \subset E$ is the* RK *of the class F_1 of all restrictions of functions of F to the subset E_1. For any such restriction $f_1 \in \mathsf{F}_1$, the norm $\|f_1\|_1$ is the minimum of $\|f\|$ for all $f \in \mathsf{F}$ whose restriction to E_1 is f_1.*

We emphasize here that the norm $\|f_1\|_1$ in the above theorem is attained at some vector $f' \in \mathsf{F}$ (see [**1**, page 351]):

$$\|f_1\|_1 = \|f'\| \text{ for some } f' \in \mathsf{F} \text{ whose restriction to } E_1 \text{ is } f_1. \tag{A.3}$$

Next we discuss the functional completion problem. Let F be a space of functions on a set E which is a pre-Hilbert space. By a functional completion of F, we mean a completion of F by adjunction of functions on E such that the evaluation map at any point $y \in E$ is a continuous function on the completed space [**1**, page 347]. Aronszajn has proved that in order that F admit a functional completion it is necessary and sufficient that

1° for every $y \in E$, the linear functional $f(y)$ defined in F be continuous;

2° for a Cauchy sequence $\{f_n\} \subseteq \mathsf{F}$, the condition $f_n(y) \to 0$ for every y implies that f_n itself converges to 0 in norm.

We notice that if the condition 1° is satisfied but 2° is not fulfilled, there always exists an abstract way so that 2° is satisfied. One way introduced in [**1**] is as follows. We first consider an abstract completion $\overline{\mathsf{F}}$ of F. We choose an additional set E' of ideal elements from $\overline{\mathsf{F}}$ such that the functions of F, extended to $E + E'$ with the same norm as in F, satisfy 2° admitting a functional completion (see [**1**, page 350] for the details).

Let us come back to our discussion. We show that the space F_0 satisfies the condition 1°.

Lemma A.2. *The space F_0 of functions on E_0 equipped with the norm $\|\cdot\|_0$ satisfies the condition 1°.*

Proof. Let $f \in \mathsf{F}_0$, and let $f_n \in \mathsf{F}_n$ be any element such that $f_{n0} = f$. Then for any fixed $y \in E_0$, by the same computation as in (A.1) we get

$$|f(y)| \leq \Big(\lim_{k\to\infty} \|f_{nk}\|_k\Big) K_0(y,y)^{1/2}.$$

Since this holds for any $f_n \in \mathsf{F}_n$ with $f_{n0} = f$, we conclude that

$$|f(y)| \le \|f\|_0 K_0(y,y)^{1/2}, \tag{A.4}$$

that is, for each $y \in E_0$ the functional $f(y)$ is continuous on F_0. □

Since the condition 2° is not assured, in general we rely on the method sketched above. By adjoining an extra set E' to E_0, if necessary, and extending the functions of F_0 to the set $E_0 + E'$ with the same norm as in F_0, we get a functionally completed space, say $\widetilde{\mathsf{F}}_0$, with an RK $\widetilde{K}_0$. Then, by using Theorem A.1, we restrict the functions to the set E_0 to get an RKHS F_0^* on E_0 with an RK K_0^*, which is the restriction of $\widetilde{K}_0$ to E_0. Because each element of $\widetilde{\mathsf{F}}_0$ is an element of $\overline{\mathsf{F}_0}$ represented as a function on the set E_0+E', to any $\widetilde{f}_0 \in \widetilde{\mathsf{F}_0}$, there corresponds a Cauchy sequence $\{f_0^{(n)}\} \subset \mathsf{F}_0$ which converges to $\widetilde{f}_0$. Moreover, by the way the norm is defined for the restrictions in Theorem A.1, the restricted class F_0^* and its norm $\|\cdot\|_0^*$ can be described in the following way (see [**1**, page 366]): $f_0^* \in \mathsf{F}_0^*$ if there is a Cauchy sequence $\{f_0^{(n)}\} \subset \mathsf{F}_0$ such that

$$f_0^*(x) = \lim_{n\to\infty} f_0^{(n)}(x) \quad \text{for every } x \in E_0, \tag{A.5}$$

and the norm is given by

$$\|f_0^*\|_0^* = \min \lim_{n\to\infty} \|f_0^{(n)}\|_0, \tag{A.6}$$

the minimum being taken for all Cauchy sequences $\{f_0^{(n)}\} \subset \mathsf{F}_0$ satisfying (A.5). By the property (A.3) there exists at least one Cauchy sequence for which the minimum is attained. Such sequences are called determining f_0^*, as in [**1**]. The scalar product corresponding to $\|\cdot\|_0^*$ is defined by

$$(f_0^*, g_0^*)_0^* = \lim_{n\to\infty} (f_0^{(n)}, g_0^{(n)})_0 \tag{A.7}$$

for any two Cauchy sequences $\{f_0^{(n)}\}$ and $\{g_0^{(n)}\}$ determining f_0^* and g_0^*. As was noted in [**1**], we remark that even when only one of $\{f_0^{(n)}\}$ or $\{g_0^{(n)}\}$ is determining, the relation (A.7) is still valid. We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. First we show that $\{K_{n0}(\cdot,y)\}$ is a Cauchy sequence in F_0. For any fixed $y \in E_0$ and $n \le m \le k$, we have from (2.4)

(A.8)
$$\begin{aligned}\|K_{mk}(\cdot,y)-K_{nk}(\cdot,y)\|_k^2 &\le \|K_m(\cdot,y)-K_{nm}(\cdot,y)\|_m^2\\ &= K_m(y,y)-\overline{K_{nm}(y,y)}-K_{nm}(y,y)\\ &\quad+\|K_{nm}(\cdot,y)\|_m^2\\ &\le K_m(y,y)-2K_n(y,y)+\|K_n(\cdot,y)\|_n^2\\ &= K_m(y,y)-K_n(y,y).\end{aligned}$$

Taking $k \to \infty$, we have

(A.9)
$$\left(\lim_{k\to\infty}\|K_{mk}(\cdot,y)-K_{nk}(\cdot,y)\|_k^2\right) \le K_m(y,y)-K_n(y,y),$$

and hence we get

(A.10)
$$\|K_{m0}(\cdot,y)-K_{n0}(\cdot,y)\|_0^2 \le K_m(y,y)-K_n(y,y),$$

because of the definition of the norm $\|\cdot\|_0$ in (2.7). This proves, together with (2.6), that $\{K_{n0}(\cdot,y)\}$ is a Cauchy sequence in F_0. By Lemma A.2, for every $x \in E_0$ the sequence $\{K_{n0}(x,y)\}$ converges to a function $K_0^*(x,y)$. By (A.5) the function $K_0^*(\cdot,y)$ belongs to K_0^*.

It remains to prove the reproducing property of K_0^*. For this, take any $f_0^* \in \mathsf{F}_0^*$ and a Cauchy sequence $\{f_0^{(n)}\} \subset \mathsf{F}_0$ determining f_0^*. For each $n \ge 1$, we can find a number $k_n \in \mathbf{N}$ and a function $f_{k_n} \in \mathsf{F}_{n_k}$ such that $f_{k_n0} = f_0^{(n)}$ and

(A.11)
$$\lim_{l\to\infty}\|f_{k_nl}\|_l^2 \le \|f_0^{(n)}\|_0^2 + \frac{1}{2n^2}.$$

Now find a number $m_n > k_n$ such that

(A.12)
$$\|f_{k_nm_n}\|_{m_n}^2 - \lim_{l\to\infty}\|f_{k_nl}\|_l^2 \le \frac{1}{2n^2}.$$

Then from (A.11) and (A.12) we have

(A.13)
$$\|f_{k_nm_n}\|_{m_n}^2 - \|f_0^{(n)}\|_0^2 \le \frac{1}{n^2}, \quad m_n > k_n.$$

It is clear that $\{K_{m_n0}(\cdot,y)\}$ is also a Cauchy sequence in F_0 such that $K_{m_n0}(x,y)$ converges to $K_0^*(x,y)$ for all $x\in E_0$. Consequently, from (A.7) it follows that

$$(K_0(\cdot,y)^*, f_0^*)_0^* = \lim_{n\to\infty}(K_{m_n0}(\cdot,y), f_0^{(n)})_0. \tag{A.14}$$

We may now write

$$\begin{aligned}(K_{m_n0}(\cdot,y), f_0^{(n)})_0 &= (K_{m_n}(\cdot,y), f_{k_nm_n})_{m_n}\\ &\qquad - \Big[(K_{m_n}(\cdot,y), f_{k_nm_n})_{m_n}\\ &\qquad\quad - (K_{m_n0}(\cdot,y), f_{k_n0})_0\Big],\end{aligned} \tag{A.15}$$

noticing that $f_{k_n0}=f_0^{(n)}$. The square bracket is of the form $[(g,h)_{m_n}-(g_0,h_0)_0]$ for g,h of F_{m_n} (g_0 and h_0 are restrictions of g and h to E_0, respectively). This is a bilinear form in g and h, and the corresponding quadratic form $(g,g)_{m_n}-(g_0,g_0)_0=\|g\|_{m_n}^2-\|g_0\|_0^2$ is positive (see (2.7)). Consequently by using the Cauchy-Schwarz inequality and (A.13) we get in connection with (A.15)

$$\begin{aligned}|[\cdots]| &\le \left[\|K_{m_n}(\cdot,y)\|_{m_n}^2 - \|K_{m_n0}(\cdot,y)\|_0^2\right]^{1/2}\\ &\quad\times\left[\|f_{k_nm_n}\|_{m_n}^2 - \|f_{k_n0}\|_0^2\right]^{1/2}\\ &\le \frac{1}{n}\|K_{m_n}(\cdot,y)\|_{m_n} = \frac{1}{n}K_{m_n}(y,y)^{1/2}.\end{aligned} \tag{A.16}$$

As $n\to\infty$ this converges to 0, since $K_{m_n}(y,y)\nearrow K_0(y,y)<\infty$. Therefore from (A.14)–(A.16) we get

$$\begin{aligned}(K_0^*(\cdot,y), f_0^*)_0^* &= \lim_{n\to\infty}(K_{m_n}(\cdot,y), f_{k_nm_n})_{m_n}\\ &= \lim_{n\to\infty} f_{k_nm_n}(y)\\ &= \lim_{n\to\infty} f_{k_n0}(y)\\ &= \lim_{n\to\infty} f_0^{(n)}(y) = f_0^*(y),\end{aligned} \tag{A.17}$$

which is the reproducing property of K_0^*. The proof is completed. □

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